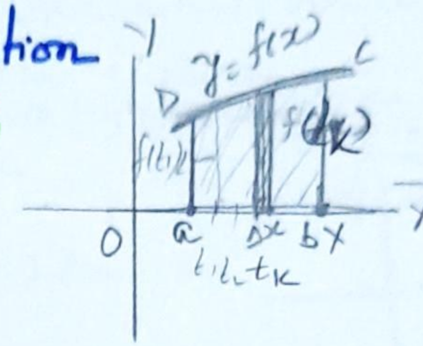
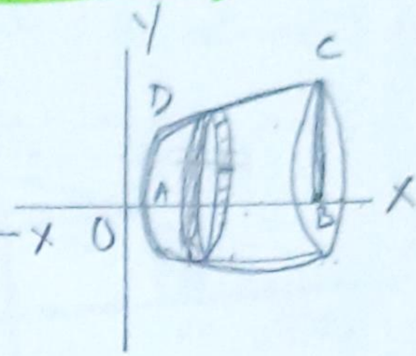
UNIT # 03/II

VOLUME OF SOLID OF REVOLUTION

Volume of solid of revolution By CIRCULAR DISKS





Let $y = f(x)$ be a continuous function s.t. $f(x) \geq 0$ in $[a, b]$

When this curve is revolved about x-axis, then it generates **solid of revolution.**

Now, we wish to find volume of solid of revolution

Divide interval $[a, b]$ in n equal parts. $\pi r^2 h$

Suppose width is $\Delta x_1, \Delta x_2, \ldots \Delta x_n$

When each strip revolve around x-axis, right circular cylinders are formed with radius $f(t_1), f(t_2) \ldots f(t_n)$

$\therefore$ volume of each cylinder is $\pi [f(t_1)]^2 \Delta x_1, \pi [f(t_2)]^2 \Delta x_2, \ldots$

$\therefore$ volume of solid of revolution $V = \sum_{k=1}^{n} \pi [f(t_k)]^2 \Delta x_k$

as $\Delta x_i \to 0$, then

$$V = \pi \int_a^b [f(x)]^2 dx \Rightarrow V = \pi \int_a^b y^2 dx$$

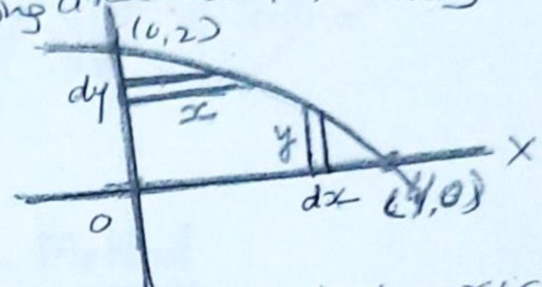
Similarly area bounded by $x = f(y)$ from $y = c$ to $y = d$, revolved about y-axis, then

$$V = \pi \int_c^d x^2 dy$$

Exp. The area bounded by the curve $y=\sqrt{4-x}$, the line $x=0$ and the line $y=0$ is revolved about the x-axis. Find the volume of the solid thus generated. Also find the volume if the area is revolved about the y-axis.

Sol. Volume of Solid made by revolving area about x-axis

Volume $V = \pi \int_0^4 y^2 dx$



$= \pi \int_0^4 (4-x)\, dx$

$= 8\pi$

Volume of solid made by revolving area about y-axis

$V = \pi \int_0^2 x^2 dy$

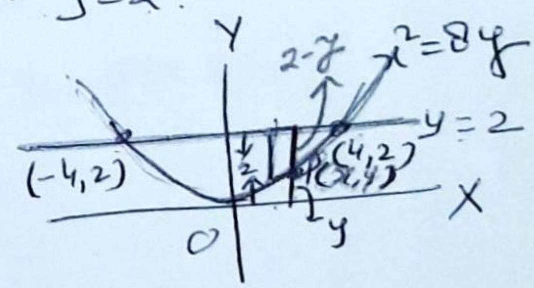
$y=\sqrt{4-x}$
$y^2 = 4-x$
$x = 4-y^2$

$= \pi \int_0^2 (4-y^2)^2 dy$

$= \frac{256\pi}{15}$ Ans

Exp. The area bounded by the parabola $x^2=8y$ and the line $y=2$ is revolved about the line $y=2$.

Sol $V = \int_{-4}^{4} \pi (2-y)^2 dx$

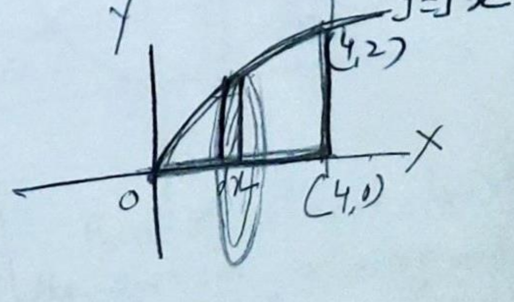


$= \int_{-4}^{4} \pi \left[2-\frac{x^2}{8}\right]^2 dx$

$= \frac{256\pi}{15}$ Ans

Exp. The region between the curve $y=\sqrt{x}$, $\boxed{0 \le x \le 4}$ and the x-axis is revolved about x-axis to generate a solid. Find it's volume.

Sol. $V = \pi \int_0^4 y^2 dx = \pi \int_0^4 x\, dx$



$= 8\pi$ Ans

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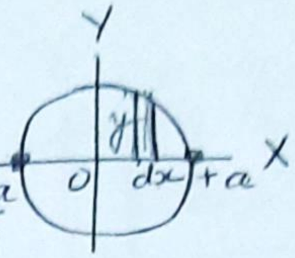
Exp: The circle $x^2+y^2=a^2$ is rotated about x-axis to generate a sphere. Find its volume.

Solution

$$V = \pi \int_{-a}^{a} y^2 dx$$

$$= \pi \int_{-a}^{a} (a^2 - x^2) dx = 2\pi \int_{0}^{a} (a^2-x^2) dx$$



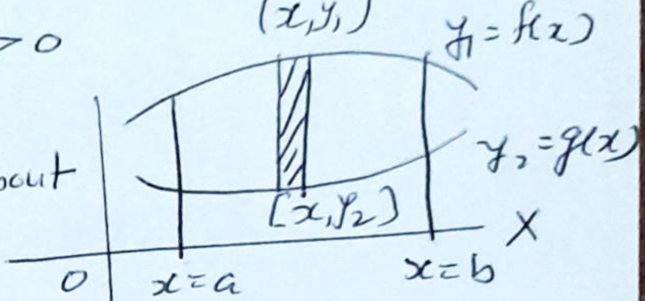
$$= \frac{4}{3}\pi a^3 \text{ Ans}$$

Volume by washer Method

Let $y_1 = f(x)$ and $y_2 = g(x)$ be continuous functions

s.t. $f(x) > g(x) > 0$

Suppose area bounded by the curves and $x=a$ and $x=b$ is revolved about X-axis.



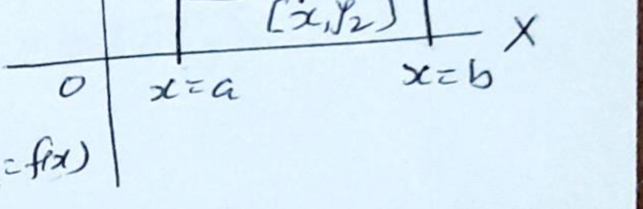
Then volume generated by the area = volume generated by $y_1 = f(x)$ − volume generated by $y_2 = g(x)$.

$$V = \pi \int_a^b y_1^2 dx - \pi \int_a^b y_2^2 dx$$

$$= \pi \int_a^b (y_1^2 - y_2^2) dx.$$

Exp. Find the volume generated by revolving about the x-axis the area bounded by parabolas. $x^2 = 2y$ and $x^2 = 12-4y$

$$V = \pi \int_{-2}^{2} (y_1^2 - y_2^2) dx$$



$$= \pi \int_{-2}^{2} \left[\left(3 - \frac{x^2}{4}\right)^2 - \left(\frac{x^2}{2}\right)^2\right] dx$$

$$= 2\pi \int_{0}^{2} \left[\left(3 - \frac{x^2}{4}\right)^2 - \left(\frac{x^2}{2}\right)^2\right] dx$$

$$= \frac{128\pi}{5} \text{ Ans}$$

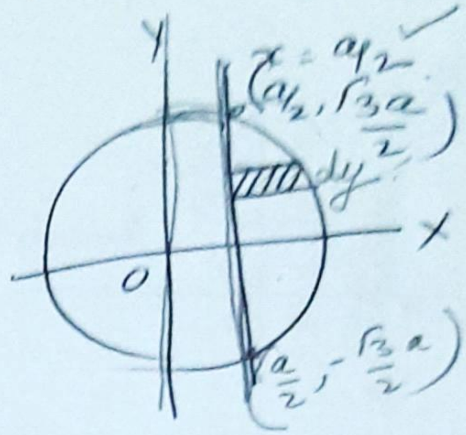
#03/II/03

Point of intersection of the curve. $x^2 = 2y$ and $x^2 = 12-4y$

$2y = 12 - 4y \Rightarrow y = 2.$

$x^2 = 2y \Rightarrow x = \pm 2.$

Exp: The smaller area bounded by $x^2+y^2=a^2$ and $x=\frac{a}{2}$, is revolved about the y-axis. Find the volume of the solid thus generated.



Solution. First, we find point of intersection of both curves

$x=a/2$ and $x^2+y^2=a^2$

$$\frac{a^2}{4}+y^2=a^2$$

$$y=\pm\frac{\sqrt{3}a}{2}$$

Point of intersection are

$$\left(\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right) \text{ and } \left(a/2, \sqrt{3}a/2\right)$$

$$-\frac{\sqrt{3}.a}{2}\le y\le\frac{\sqrt{3}a}{2}$$

upper curve x_1 is $\underline{x^2+y^2=a^2} \Rightarrow x=\pm\sqrt{a^2-y^2}$

lower curve x_2 is $x=a/2$

$$V=\pi\int_{-\frac{\sqrt{3}a}{2}}^{\frac{\sqrt{3}a}{2}}(x_1^2-x_2^2)\,dy$$

$$=2\pi\int_0^{\frac{\sqrt{3}a}{2}}\left[(a^2-y^2)-\frac{a^2}{4}\right]dy$$

$$=2\pi\int_0^{\frac{\sqrt{3}a}{2}}\left[\frac{3a^2}{4}-y^2\right]dy$$

$$=\frac{\sqrt{3}\pi a^3}{2}$$ **Ans.**

Volume by cylindrical shell Method

Suppose area bounded by

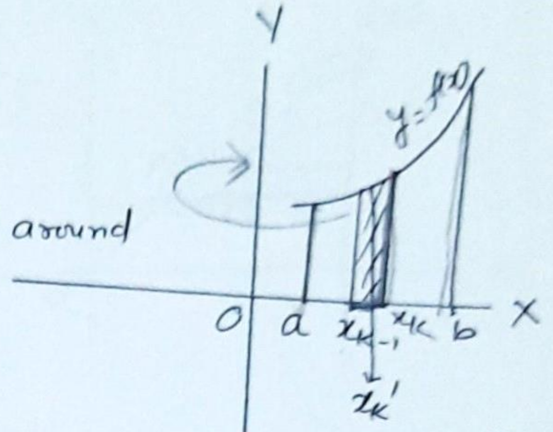
$y = f(x)$, $x = a$, $x = b$

is revolved around y-axis.

Take n rectangles from a to b

When these rectangles are revolved around y-axis a hollow shell will be formed,

Then volume of one shell say k^{th} shell will be



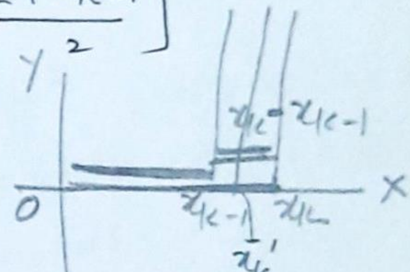
$$\Delta V_k = \pi(x_k^2 - x_{k-1}^2) f(x_k') \qquad [\text{volume of cylinder} = \pi r^2 h]$$

$$= 2\pi \frac{(x_k + x_{k-1})}{2}(x_k - x_{k-1}) f(x_k')$$

$$= 2\pi x_k'' \Delta x_k f(x_k') \qquad \left[x_k'' = \frac{x_k + x_{k-1}}{2}\right]$$

$\therefore$ Total volume

$$V_n = \sum_{k=1}^{n} 2\pi x_k'' f(x_k') \Delta x_k$$



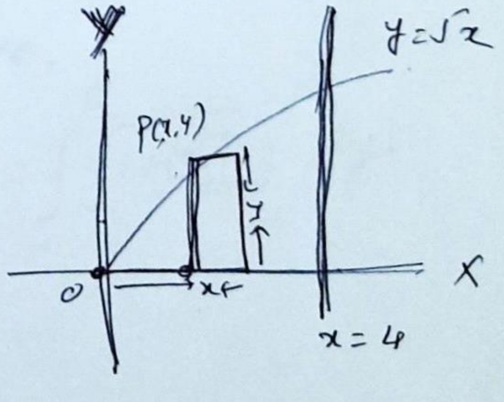
as $n \to \infty$,

$$V = 2\pi \int_a^b x f(x)\, dx \quad \text{or} \quad \boxed{V = 2\pi \int_a^b xy\, dx}$$

Exp The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$, is revolved about y-axis to generate a solid. Find the volume of the solid by shell method.

Sol.

$$V = 2\pi \int_0^4 xy\, dx$$



$$= 2\pi \int_0^4 x\sqrt{x}\, dx$$

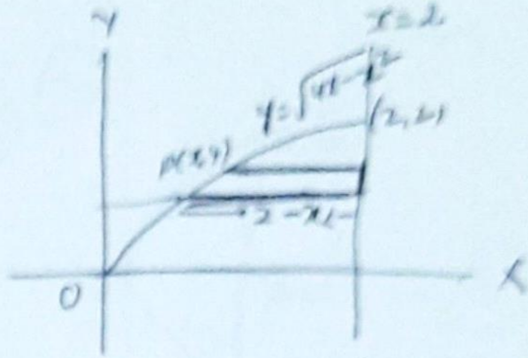
$$= 2\pi \int_0^4 x^{1+\frac{1}{2}} dx = 2\pi \int_0^4 x^{3/2} dx$$

$$= \frac{128\pi}{5} \quad \underline{\text{Ans}}$$

#03/II/05

Exp. The region bounded by the curve $y = \sqrt{4x - x^2}$, the x-axis and the line $x = 2$ is revolved about the x-axis. Find the volume of the solid by cylindrical shell method.

Sol.



$$y = \sqrt{4x - x^2}$$
$$y^2 = 4x - x^2$$
$$y^2 - 4x + x^2 = 0$$
$$x^2 - 4x + 4 - 4 + y^2 = 0$$
$$(x-2)^2 + y^2 = 4$$
$$\Rightarrow (x-2)^2 = 4 - y^2$$
$$\Rightarrow (2-x)^2 = 4 - y^2$$
$$\Rightarrow 2 - x = \sqrt{4 - y^2}$$

$$V = 2\pi \int_0^2 y(2-x)\,dy$$
$$= 2\pi \int_0^2 y\sqrt{4-y^2}\,dy$$

, suppose $4 - y^2 = t^2$, At $y = 0$, $t = 2$
$-\cancel{2}y\,dy = \cancel{2}t\,dt$, At $y = 2$, $t = 0$
$y\,dy = -t\,dt$

$$V = -2\pi \int_2^0 t.t\ dt$$
$$= 2\pi \int_0^2 t^2 dt = \frac{16\pi}{3} \text{ Ans}$$

RECTIFICATION

Computing length of the arc of curve.

$$\overset{\frown}{AB} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\cdot dx \qquad \overset{\frown}{AB} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\cdot dy$$

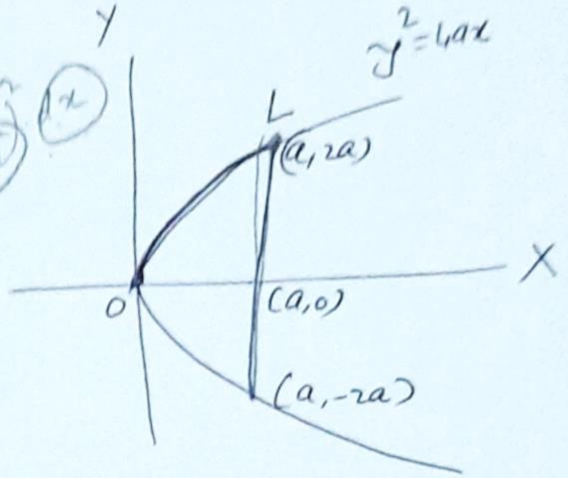
In polar form

$$\overset{\frown}{AB} = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\cdot d\theta$$

Exp. Find the length of the arc of the parabola $y^2=4ax$, $(a>0)$ measured from the vertex to one extremity of it's latus rectum.

Sol.
$$\widehat{OL}=\int_0^{2a}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad \text{---} (1)$$

$\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$



$y^2=4ax$

$\Rightarrow x=\frac{y^2}{4a}$

$\Rightarrow \frac{dx}{dy}=\frac{2y}{4a}=\frac{y}{2a}$

$\therefore$ From eq (1)

$$\widehat{OL}=\int_0^{2a}\sqrt{1+\frac{y^2}{4a^2}}\,dy$$

$$=\int_0^{2a}\frac{1}{2a}\sqrt{y^2+4a^2}\,dy$$

$$=\frac{1}{2a}\int_0^{2a}\sqrt{y^2+4a^2}\,dy$$

$$=\frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2}+\frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{2a}$$

$$=\frac{1}{2a}\left[a\sqrt{8a^2}+2a^2\log\left(2a+\sqrt{8a^2}\right)-2a^2\log 2a\right]$$

$$=a\left[\sqrt{2}+\log(1+\sqrt{2})\right] \text{ Ans}$$

$\because y^2=4ax$

$\Rightarrow \frac{dy}{dx}=\frac{4a}{2y}=\frac{2a}{y}$

$\frac{dx}{dy}$

$\sqrt{1+\frac{4a^2}{y^2}}$

$x=\frac{y^2}{4a}$

$\frac{dx}{dy}=\frac{2y}{4a}=\frac{y}{2a}$

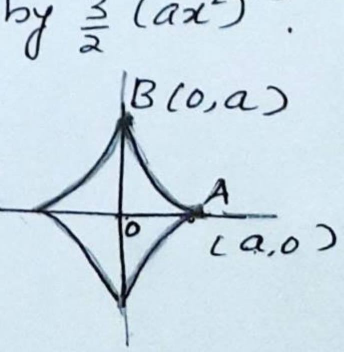
Exp (A) Find the entire length of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

(B) Prove that the length of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ measured from $(0,a)$ to the point (x,y) is given by $\frac{3}{2}(ax^2)^{1/3}$.

Sol. (A)

Entire length of the astroid is

$$4\times\widehat{AB}=4\int_a^0\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \text{---} (1)$$



Now, $x^{2/3}+y^{2/3}=a^{2/3}$

Differentiating w.r.t. x, we get

#03/7/07

$$\frac{2}{3}x^{2/3-1} + \frac{2}{3}y^{2/3-1}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2}{3}\left[x^{-1/3} + y^{-1/3}\frac{dy}{dx}\right] = 0$$

$$\Rightarrow x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow y^{-1/3}\frac{dy}{dx} = -x^{-1/3}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}}$$

From eq. (1)

$$\text{Arc length} = 4\int_a^0 \sqrt{1+\left(\frac{y^{1/3}}{x^{1/3}}\right)^2}\cdot dx$$

$$= 4\int_a^0 \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\, dx$$

$$= 4\int_a^0 \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\cdot dx$$

$$= 4\int_a^0 \sqrt{\frac{a^{2/3}}{x^{2/3}}}\cdot dx = 4\int_a^0 \frac{a^{1/3}}{x^{1/3}}\, dx$$

$$= 4a^{1/3}\int_a^0 x^{-1/3}\, dx$$

$$= -6a.$$

Arc length is always +ve, so entire arc length of the astroid $= 6a$. Ans

(B) The required length is $\displaystyle\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx.$

$$= a^{1/3}\int_0^x x^{-1/3}\, dx.$$

$$= a^{1/3}\left[\frac{3x^{2/3}}{2}\right]_0^x$$

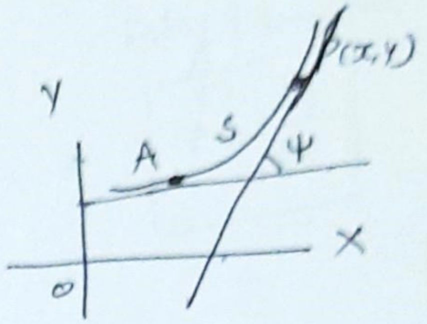
$$= \frac{3}{2}a^{1/3}x^{2/3}$$

$$= \frac{3}{2}(ax^2)^{1/3} \quad \text{Ans}$$

INTRINSIC EQUATION

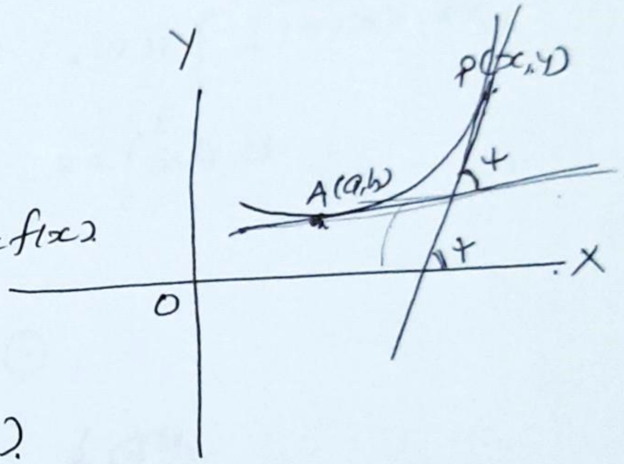
DEFINITION

Let A be any fixed point on the curve and $P(x,y)$ is any point on it. Let ψ be a angle between the tangents at A and P and S is the arc AP. Then the relation between S and ψ is called intrinsic equation of the curve.



Thm. Obtain the intrinsic eqs of the curve represented in.
(i) CARTESIAN FORM
(ii) POLAR FORM.

Proof. (i) CARTESIAN FORM



Let $A(a,b)$ be a fixed point and $P(x,y)$ is any point on the curve $y=f(x)$. We assume here that tangent at A is $\parallel$ to X-axis.

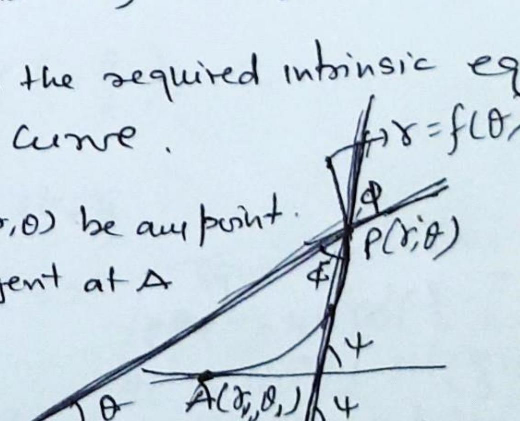
$$S = \overset{\frown}{AP} = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad —(1)$$

and $\tan\psi = \dfrac{dy}{dx}$ —(2)

Eliminating x from eqs (1) and (2), we get a relation

$F(S,\psi)=0$, which is the required intrinsic eq. of the curve.

(ii) POLAR FORM.



Let $A(r_1,\theta_1)$ be a fixed point and $P(r,\theta)$ be any point on the curve $r=f(\theta)$. Assume that tangent at A is $\parallel$ to polar axis.

$$S = \int_{\theta_1}^{\theta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad —(1)$$

—(2)
We know that $\psi = \theta + \phi$, ϕ is the angle between radius vector and tangent at P

$\tan\phi = r\dfrac{d\theta}{dr}$ —(3)

Eliminating ϕ and θ from eqs (1), (2) and (3), we get $F(S,\psi)=0$. which is the intrinsic equation.

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Exp. Find the intrinsic equation of the ~~cardioide~~ cardioid, $r = a(1+\cos\theta)$.
Hence prove that $s^2 + 9\rho^2 = 16a^2$, where ρ is the radius of curvature at any point of the curve.

Radius of curvature
$$\boxed{\rho = \frac{ds}{d\psi}}$$

Sol.
$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{—(1)}$$

$r = a(1+\cos\theta)$, $\therefore \frac{dr}{d\theta} = -a\sin\theta$.

Hence from eq (1)
$$s = \int_0^\theta \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\; d\theta$$
$$= a\int_0^\theta \sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}\; d\theta$$
$$= a\int_0^\theta \sqrt{2+2\cos\theta}\; d\theta = a\sqrt{2}\int_0^\theta \sqrt{1+\cos\theta}\; d\theta$$
$$= a\sqrt{2}\int_0^\theta \sqrt{2\cos^2\theta/2}\; d\theta = 2a\int_0^\theta \cos\theta/2\; d\theta$$
$$= 2a\left[\sin\theta/2\right]_0^\theta \cdot 2$$
$$s = 4a\sin\theta/2 \quad \text{—①}$$

We know that $\psi = \theta + \phi$ —(2)
$$\tan\phi = r\frac{d\theta}{dr} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta} = -\frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}$$
$$= -\frac{\cos\theta/2}{\sin\theta/2}$$
$$= -\cot\theta/2 = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$
$$\therefore \phi = \frac{\pi}{2} + \theta/2$$

From eq (2). $\psi = \theta + \frac{\pi}{2} + \theta/2 = \frac{3\theta}{2} + \frac{\pi}{2}$.
$$\Rightarrow \theta/2 = \frac{\psi}{3} - \frac{\pi}{6} \quad \text{—(3)}$$

from eq ① & ③, we get
$$s = 4a\sin\left(\frac{\psi}{3} - \frac{\pi}{6}\right) \quad \text{Ans}$$

Now,
$$\rho = \frac{ds}{d\psi} = 4a\cos\left(\frac{\psi}{3} - \frac{\pi}{6}\right) \times \frac{1}{3}$$
$$= \frac{4a}{3}\cos\left(\frac{\psi}{3} - \frac{\pi}{6}\right)$$
$$3\rho = 4a\cos\left(\frac{\psi}{3} - \frac{\pi}{6}\right)$$

squaring both sides
$$9\rho^2 = 16a^2\cos^2\left(\frac{\psi}{3} - \frac{\pi}{6}\right)$$
$$9\rho^2 = 16a^2\left[1 - \sin^2\left(\frac{\psi}{3} - \frac{\pi}{6}\right)\right]$$
$$= 16a^2 - 16a^2\sin^2\left(\frac{\psi}{3} - \frac{\pi}{6}\right)$$
$$= 16a^2 - s^2$$
$$s^2 + 9\rho^2 = 16a^2$$
Proved

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Exp. Show that the intrinsic equation of the curve $y^3 = ax^2$ is
$$27s = 8a(\sec^3\psi - 1)$$

Sol.
$$s = \int_0^y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad \text{---(1)}$$

Given curve is $y^3 = ax^2$
$$\Rightarrow x^2 = \frac{y^3}{a}$$
$$\Rightarrow x = \frac{y^{3/2}}{\sqrt{a}}$$
$$\therefore \frac{dx}{dy} = \frac{1}{\sqrt{a}}\cdot\frac{3}{2}y^{\frac{3}{2}-1} = \frac{3}{2\sqrt{a}}y^{1/2} = \frac{3}{2}\sqrt{\frac{y}{a}}$$

$$\tan\psi = \frac{dx}{dy} \Rightarrow \tan\psi = \frac{3}{2}\sqrt{\frac{y}{a}}$$

From eq. (1) $s = \int_0^y \sqrt{1+\frac{9}{4}\frac{y}{a}}\cdot dy = \int_0^y \left(1+\frac{9y}{4a}\right)^{1/2} dy$
$$= \frac{4a}{9}\cdot\frac{2}{3}\left(1+\frac{9y}{4a}\right)^{3/2}\Big|_0^y$$
$$s = \frac{8a}{27}\left[\left(1+\frac{9y}{4a}\right)^{3/2} - 1\right]$$
$$27s = 8a\left[\left(1+\tan^2\psi\right)^{3/2} - 1\right]$$
$$= 8a\left(\left(\sec^2\psi\right)^{3/2} - 1\right)$$
$$27s = 8a(\sec^3\psi - 1) \quad \text{Ans}$$

AREA OF SURFACE OF REVOLUTION

FOR CARTESIAN CURVES

Surface area of revolution about X-axis $y = f(x)$ $\quad S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

" " " Y-axis $x = g(y)$ $\quad S = \int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

FOR PARAMETERIC CURVES

$x = f(t),\ y = g(t)\ a \le t \le b$

Revolution about X-axis $(y \geq 0)$ $\quad S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Revolution about Y-axis $(x \geq 0)$ $\quad S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

#03/II/11

Exp The arc of the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ is revolved about x - axis. Find the area of the surface thus generated.

Solution

$$S = \int_0^{\pi/2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{——— (1)}$$

Here $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$

$\therefore$ from eq (1) $S = 2\pi \int_0^{\pi/2} \cos x \sqrt{1+\sin^2 x}\,.\, dx$

Suppose $\sin x = t \Rightarrow \cos x\, dx = dt$

At $x = 0$, $t = 0$

At $x = \pi/2$, $t = 1$

Formula only

$$\int \sqrt{1+x^2}\, dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left(1+\sqrt{1+x^2}\right)$$

$$\therefore S = 2\pi \int_0^1 \sqrt{1+t^2}\,.\, dt$$

$$= 2\pi \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(1+\sqrt{1+t^2}\right)\right]_0^1$$

$$= 2\pi \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2}) - \frac{1}{2}\log(1+1)\right]$$

$$= 2\pi \left[\frac{1}{\sqrt{2}} + \frac{1}{2}\log\frac{1+\sqrt{2}}{2}\right]$$

$$= \pi\left[\sqrt{2} + \log\left(\frac{1+\sqrt{2}}{2}\right)\right] \quad \text{Ans.}$$

Exp. Find the area of the surface generated by revolving about the y-axis of the parabola $x = 2\sqrt{y}$ from $y = 0$ to $y = 3$.

Solution

$$S = \int_0^3 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Here $x = 2\sqrt{y}$

$$\therefore \frac{dx}{dy} = \cancel{2}\,.\,\frac{1}{\cancel{2}}\, y^{\frac{1}{2}-1} = \frac{1}{\sqrt{y}}$$

$$\therefore S = 2\pi \int_0^3 2\sqrt{y}\,.\sqrt{1+\frac{1}{y}}\,.\, dy$$

$$S = 4\pi \int_0^3 \sqrt{y}\sqrt{\frac{y+1}{y}} \cdot dy$$

$$= 4\pi \int_0^3 \sqrt{1+y} \cdot dy$$

$$= 4\pi \frac{2}{3}\left[(1+y)^{3/2}\right]_0^3$$

$$= \frac{8\pi}{3}\left[(1+3)^{3/2} - 1\right]$$

$$= \frac{8\pi}{3}\left[2^{2\times 3/2} - 1\right]$$

$$= \frac{8\pi}{3}(8-1)$$

$$= \frac{56\pi}{3} \quad \underline{\text{Ans}}$$

$(1+y)^{1/2}$

$\frac{(1+y)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$\frac{2}{3}(1+y)^{3/2}$

Exp Find the area of the surface generated by revolving the curve. $y = 2\sqrt{x}$, $1 \le x \le 2$ about x-axis

Sol.

$$S = \int_1^2 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

Here $y = 2\sqrt{x} \Rightarrow \frac{dy}{dx} = \cancel{2} \times \frac{1}{\cancel{2}\sqrt{x}} = \frac{1}{\sqrt{x}}$

$$S = 2\pi \int_1^2 2\sqrt{x}\sqrt{1+\frac{1}{x}} \cdot dx$$

$$= 4\pi \int_1^2 \sqrt{1+x} \; dx$$

$$= 4\pi \left[\frac{2}{3}(1+x)^{3/2}\right]_1^2$$

$$= \frac{8\pi}{3}\left[3^{3/2} - 2^{3/2}\right] \quad \underline{\text{Ans.}}$$

Exp. The line $x = 1-y$, $0 \le y \le 1$; is revolved about the y-axis to generate the cone. Find it's lateral surface area

Sol. $S = \int_0^1 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$

$x = 1 - y$

$\frac{dx}{dy} = -1$

$\therefore \; S = 2\pi \int_0^1 (1-y)\sqrt{1+1}\, dy$

$= 2\sqrt{2}\,.\pi \int_0^1 (1-y)\, dy$

$= \sqrt{2}\,\pi$ **Ans.**

Exp Find the area of the surface generated by revolving the curve $x = a\cos\theta$, $y = a\sin\theta$ from $\theta = 0$ to $\theta = \pi$ about X-axis.

Solution $S = \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

Here $x = a\cos\theta$

$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta$ and

$y = a\sin\theta$

$\frac{dy}{d\theta} = a\cos\theta$

$S = 2\pi \int_0^{\pi} a\sin\theta \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\, . d\theta$

$= 2\pi a^2 \int_0^{\pi} \sin\theta \sqrt{\sin^2\theta + \cos^2\theta}\, . d\theta$

$= 2\pi a^2 \int_0^{\pi} \sin\theta\, d\theta$

$= 2\pi a^2 . (-\cos\theta)_0^{\pi}$

$= -2\pi a^2 (\cos\pi - \cos 0)$

$= -2\pi a^2 (-1-1)$

$= 4\pi a^2$. **Ans.**

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